A study of random walks on wedges

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September 17, 2012

Abstract

In this paper we develop the idea of Lyons and gives a simple criterion for the recurrence and the transience. We also show that a wedge has the infinite collision property if and only if it is a recurrent graph.

2000 MR subject classification: 60K

Key words: random walk, wedge, infinite collision property, recurrence, resistance

1 Introduction

Let us recall briefly the definition of a wedge of $\mathbb{Z}^{d+1}$. Let $f_1, \cdots, f_d$ be a collection of d increasing functions from $\mathbb{Z}^+$ to $\mathbb{R}^+ \cup \{+\infty\}$. They induces a wedge, $\text{Wedge}(f_1, \cdots, f_d) = (\mathbb{V}, \mathbb{E})$, which has vertex set

$$\mathbb{V} = \{(x_1, \cdots, x_d, n) \in \mathbb{Z}^{d+1} : n \geq 0,\ 0 \leq x_i \leq f_i(n) \text{ for each } 1 \leq i \leq d\}$$

and edge set

$$\mathbb{E} = \{[u, v] : \|u - v\|_1 = 1, u, v \in \mathbb{V}\}.$$

Is a wedge recurrent or transient? (A locally finite connect graph is called transient or recurrent according to the type of simple random walk on it.) Lyons[8] first give

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the result that suppose (A) holds, then Wedge$(f_1, \cdots, f_d)$ is recurrent if and only if

$$\sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{1}{f_i(n)+1} = \infty. \tag{1.1}$$

Where

(A): $f_i(n+1) - f_i(n) \in \{0, 1\}$ for all $1 \le i \le d$ and all $n \ge 0$.

Readers can refer to [1][9] for more background about wedge and the reference therein.

We develop the idea of Lyons in this paper. However, our result does not rely on the condition (A). Define d increasing integer valued functions $h_1, \cdots, h_d$. Let $h_i(0) = 0$ for each $1 \le i \le d$. For each $1 \le i \le d$ and $n \ge 1$, if $h_i(n-1) + 1 > f_i(n)$ then let

$$h_i(n) = h_i(n-1);$$

otherwise, if $h_i(n-1) + 1 \le f_i(n)$ then let

$$h_i(n) = h_i(n-1) + 1.$$

Then we have our first result.

Theorem 1.1 *Wedge*$(f_1, \cdots, f_d)$ *is recurrent if and only if*

$$\sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{1}{h_i(n)+1} = \infty. \tag{1.2}$$

Example. Suppose $d = 2$, $f_1(x) = 2^x$ and $f_2(x) = \log(x+1)$. Obviously (1.1) does not succeed. On the other hand, $h_1(n) = n$ and $h_2(n) = [\log(n+1)]$. Then (1.2) holds and Wedge(f_1, f_2) is recurrent.

Now we turn to another question. As usual, we say that a graph has the infinite collision property if two independent simple random walks on the graph will collide infinitely many times, almost surely. Likewise we say that a graph has the finite collision property if two independent simple random walks on the graph collide finitely many times almost surely. It is interesting to known whether or not a graph

Fig 1: Wedge(g) has the infinite collision property, where $g(n)=n^2$.

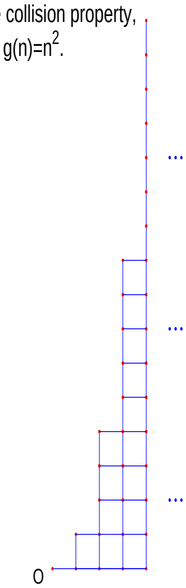
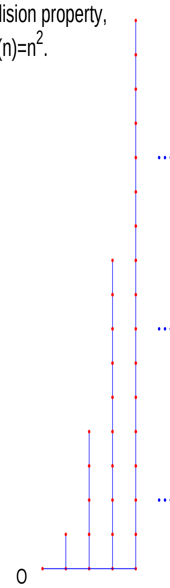


Fig 2: Comb(Z, g) has the finite collision property, where $g(n)=n^2$.



has the infinite collision property. Refer to Polya[10], Liggett[7] and Krishnapur & Peres[6] for details. To my interest is the type of a wedge. Other graphes, such as wedge combs, trees or random environment, are studied in [2][3][4][5][11] etc..

Theorem 1.2 *Wedge*$(f_1, \cdots, f_d)$ *has the infinite collision property if and only if Wedge*$(f_1, \cdots, f_d)$ *is recurrent.*

To understand the conditions better, it is worthwhile to compare a wedge with a wedge comb. Wedge(g) always has the infinite collision property since any subgraph of $\mathbb{Z}^2$ is recurrent. However, Comb$(\mathbb{Z}, g)$ may have the finite collision property [2][6]. Refer to Figure 1 and Figure 2. It implies that our theorem holds owing to the monotone property of the profile $f_i(\cdot)$ of the wedge.

2 A partition of vertex set $\mathbb{V}$

Obviously, the functions $h_1, \cdots, h_d$ defined in Section 1 satisfy that for each $1 \le i \le d$ and each $n \ge 0$,

$$0 \le h_i(n) \le f_i(n) \quad \text{and} \quad h_i(n+1) - h_i(n) \in \{0, 1\}. \tag{2.1}$$

We shall define a class of subsets $\Delta_i(n)$ and ∂_n through these functions. We shall show later that $\{\partial_n : n \ge 0\}$ is a partition of $\mathbb{V}$. For each $1 \le i \le d+1$, let

$$\Delta_i(0) = \{(0, \cdots, 0)\} \in \mathbb{Z}^{d+1}.$$

Fix $n \ge 1$, let

$$\Delta_{d+1}(n) = \{(x_1, \cdots, x_d, n) \in \mathbb{Z}^{d+1} : 0 \le x_i \le h_i(n), 1 \le i \le d\}.$$

Then $\Delta_{d+1}(n)$ is a subset of $\mathbb{V}$. Fix $1 \le i \le d$. If $h_i(n) = h_i(n-1) + 1$ then let

$$\Delta_i(n) = \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : x_j \le h_j(n) \text{ for each } 1 \le j \le d,\ x_i = h_i(n),\ x_{d+1} \le n\}.$$

Otherwise, if $h_i(n) = h_i(n-1)$ then let $\Delta_i(n) = \emptyset$.
For each $n \ge 0$ we set

$$\partial_n = \bigcup_{i=1}^{d+1} \Delta_i(n).$$

Finally, for each $x \in \mathbb{R}^{d+1}$ and each $1 \le i \le d+1$, we denote by x_i the i-th coordinate of x. For each $x \in \mathbb{V}$ and $1 \le i \le d$, we set

$$p_i(x) = \min\{m : h_i(m) \ge x_i\}.$$

By (2.1)

$$h_i(p_i(x)) = x_i.$$

For each $x \in \mathbb{V}$, set

$$u(x) = \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\}.$$

Then we have the following lemma.

Lemma 2.1 *For each pair of $m \geq 0$ and $x \in \mathbb{V}$, vertex $x \in \partial_m$ if and only if $u(x) = m$.*

Proof. Fix $x = (x_1, \cdots, x_d, n) \in \mathbb{V}$. For conciseness, we write p_i instead of $p_i(x)$. First we shall prove the statement that if $u(x) = m$ then $x \in \partial_m$. Set

$$S = \{i : 1 \leq i \leq d, x_i > h_i(n)\}.$$

We consider two cases $S = \emptyset$ and $S \neq \emptyset$.

Case I: $S = \emptyset$. Then for each $1 \leq i \leq d$,

$$x_i \leq h_i(n).$$

As a result,

$$x \in \Delta_{d+1}(n) \subset \partial_n.$$

Since $h_i(p_i) = x_i$,

$$h_i(p_i) \leq h_i(n).$$

By the definition of $p_i(\cdot)$,

$$p_i \leq n.$$

Therefore, $u(x) = n$ as claimed above.

Case II: $S \neq \emptyset$. Fix $j \in S$ which satisfies that for all $l \in S$,

$$p_l \leq p_j. \tag{2.2}$$

We shall show that $u(x) = p_j$ and $x \in \partial_{p_j}$. Since $j \in S$,

$$h_j(p_j) = x_j > h_j(n).$$

It implies that

$$n < p_j. \tag{2.3}$$

Furthermore, for each $l \in \{1, \cdots, d\} \setminus S$

$$h_l(p_l) = x_l \leq h_l(n) \leq h_l(p_j). \tag{2.4}$$

As a result of that

$$p_l \leq p_j. \tag{2.5}$$

Owing to (2.2), (2.3) and (2.5),

$$u(x) = p_j.$$

On the other hand, by the definition of $p_j(\cdot)$ there has

$$\text{either} \quad p_j = 0 \quad \text{or} \quad h_j(p_j - 1) < h_j(p_j).$$

However, there always have

$$\Delta_j(p_j) = \{(y_1, \cdots, y_d, y_{d+1}) \in \mathbb{V} : y_l \leq h_l(p_j) \text{ for each } 1 \leq l \leq d,\ y_j = h_j(p_j),\ y_{d+1} \leq p_j\}. \tag{2.6}$$

By (2.2), for each $l \in S$

$$x_l = h_l(p_l) \leq h_l(p_j). \tag{2.7}$$

By (2.4), (2.6) and (2.7), we have that

$$x \in \Delta_j(p_j) \subset \partial_{p_j}.$$

Such we have proved the first statement for both cases.

Next we shall show that $\partial_0, \partial_1, \cdots$ are disjoined. Fix $n > m \geq 0$. Since that for any $x \in \Delta_{d+1}(n)$ and any $y \in \partial_m$,

$$x_{d+1} = n > m \geq y_{d+1}.$$

So,

$$\partial_m \cap \Delta_{d+1}(n) = \emptyset. \tag{2.8}$$

Fix $1 \leq i \leq d$ and $1 \leq j \leq d$. We will show that $\Delta_i(m) \cap \Delta_j(n) = \emptyset$. Otherwise, suppose $\Delta_i(m) \cap \Delta_j(n) \neq \emptyset$. Then

$$\Delta_i(m) = \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : x_l \leq h_l(m) \text{ for each } 1 \leq l \leq d,\ x_i = h_i(m),\ x_{d+1} \leq m\},$$

$$\Delta_j(n) = \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : x_l \leq h_l(n) \text{ for each } 1 \leq l \leq d,\ x_j = h_j(n),\ x_{d+1} \leq n\}.$$

And then

$$h_j(n) = h_j(n-1) + 1.$$

Furthermore, since $\Delta_i(m) \cap \Delta_j(n) \neq \emptyset$ there exists $z \in \Delta_i(m) \cap \Delta_j(n)$. Then

$$z_j = h_j(n) \text{ and } z_l \leq \min\{h_l(m), h_l(n)\} \text{ for each } 1 \leq l \leq d.$$

Hence,

$$h_j(n) \leq h_j(m). \tag{2.9}$$

On the other hand, since $h_j(\cdot)$ is an increasing function and $n > m$,

$$h_j(n-1) \geq h_j(m).$$

It deduces that

$$h_j(n) = h_j(n-1) + 1 \geq h_j(m) + 1 > h_j(m).$$

This contradict (2.9). Therefore,

$$\Delta_i(m) \cap \Delta_j(n) = \emptyset. \tag{2.10}$$

Similarly, we can prove that

$$\Delta_i(n) \cap \Delta_{d+1}(m) = \emptyset. \tag{2.11}$$

Taking (2.8), (2.10) and (2.11) together, we get that ∂_n and ∂_m are disjoined. We have finished the proof of the lemma. □

The next lemma shows that the neighbor of ∂_n are ∂_{n-1} and ∂_n for each $n \geq 1$. It implies that ∂_n is a cutset of the graph Wedge$(f_1, \cdots, f_d)$. We write

$$e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$$

for the i-th unit vector of $\mathbb{R}^{d+1}$.

Lemma 2.2 *Let $x \in \mathbb{V}$ and $1 \leq i \leq d+1$. If $x + e_i \in \mathbb{V}$ then*

$$u(x + e_i) - u(x) = 0 \text{ or } 1.$$

Proof. Fix $x \in \mathbb{V}$. Obviously for each $1 \le i \le d+1$ and $1 \le l \le d+1$ with $i \ne l$, if $x + e_l \in \mathbb{V}$ then

$$p_i(x + e_l) = p_i(x).$$

First we consider the easy case $i = d+1$. Obviously, $x + e_{d+1} \in \mathbb{V}$. Hence

$$\begin{aligned} u(x+e_{d+1}) - u(x) =& \max\{x_{d+1}+1, p_1(x+e_{d+1}), \cdots, p_d(x+e_{d+1})\} - \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\} \\ =& \max\{x_{d+1}+1, p_1(x), \cdots, p_d(x)\} - \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\} \\ =& \ 0 \text{ or } 1. \end{aligned}$$

Next we consider the case $1 \le i \le d$. Fix $x \in \mathbb{V}$ and $x + e_i \in \mathbb{V}$.
If $f_i(p_i(x)+1) \ge x_i + 1$, then

$$f_i(p_i(x)+1) \ge x_i + 1 = h_i(p_i(x)) + 1.$$

Hence

$$h_i(p_i(x)+1) = h_i(p_i(x)) + 1 = x_i + 1.$$

Such

$$p_i(x+e_i) = p_i(x) + 1.$$

Similarly we have

$$\begin{aligned} & u(x+e_i) - u(x) \\ =& \max\{x_{d+1}, p_1(x+e_i), \cdots, p_d(x+e_i)\} - \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\} \\ =& \max\{x_{d+1}, p_1(x), \cdots, p_{i-1}(x), p_i(x)+1, p_{i+1}(x), \cdots, p_d(x)\} - \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\} \\ =& \ 0 \text{ or } 1. \end{aligned}$$

Otherwise, $f_i(p_i(x)+1) < x_i + 1$. Let

$$\eta_i = \min\{m : f_i(m) \ge x_i + 1\}.$$

Then

$$\eta_i > p_i(x) + 1.$$

Furthermore,

$$h_i(\eta_i - 1) \geq h_i(p_i(x)) = x_i.$$

On the other hand

$$h_i(\eta_i - 1) \leq f_i(\eta_i - 1) < x_i + 1.$$

Since $h_i(\cdot)$ is integer valued,

$$h_i(\eta_i - 1) = x_i.$$

As a result,

$$f_i(\eta_i) \geq x_i + 1 = h_i(\eta_i - 1) + 1.$$

Hence

$$h_i(\eta_i) = h_i(\eta_i - 1) + 1 = x_i + 1.$$

Therefore,

$$p_i(x + e_i) \leq \eta_i. \tag{2.12}$$

Since $x + e_i \in \mathbb{V}$,

$$f_i(x_{d+1}) \geq x_i + 1.$$

and then

$$\eta_i \leq x_{d+1}.$$

By (2.12),

$$p_i(x + e_i) \leq x_{d+1}.$$

So that,

$$\begin{aligned}
&u(x + e_i) - u(x) \\
=& \max\{x_{d+1}, p_1(x + e_i), \cdots, p_d(x + e_i)\} - \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\} \\
\leq& \max\{x_{d+1}, p_1(x), \cdots, p_{i-1}(x), x_{d+1}, p_{i+1}(x), \cdots, p_d(x)\} - \max\{x_{d+1}, p_1(x), \cdots, p_d(x)\} \leq 0.
\end{aligned}$$

By the increasing property of $u(\cdot)$, we get that

$$u(x + e_i) - u(x) = 0.$$

□

At the end of this section, we shall estimate the cardinality of ∂_n.

Lemma 2.3 *For each* $n \geq 0$,

$$\prod_{i=1}^{d}(h_i(n)+1) \leq |\partial_n| \leq (d+1)\prod_{i=1}^{d}(h_i(n)+1).$$

Proof. For each $n \geq 0$

$$|\partial_n| \geq |\Delta_{d+1}(n)| = \prod_{i=1}^{d}(h_i(n)+1),$$

since $\Delta_{d+1}(n) \subseteq \partial_n$.

Fix $n \geq 1$ and $1 \leq i \leq d$. Without making confusion, we set

$$p_i = p_i(n) = \min\{m : h_i(m) = n\}.$$

Then

$$\Delta_i(p_i) = \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : x_l \leq h_l(p_i) \text{ for each } 1 \leq l \leq d,\ x_i = n,\ x_{d+1} \leq p_i\}.$$

As we have known that if $x \in \mathbb{V}$ with $x_i = n$ then $f_i(x_{d+1}) \geq n$. Let

$$k = \min\{u \in \mathbb{Z}^+ : f_i(u) \geq n\}.$$

Then

$$\begin{aligned}\Delta_i(p_i) =& \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : 0 \leq x_l \leq h_l(p_i) \text{ for each } 1 \leq l \leq d,\ x_i = n,\ k \leq x_{d+1} \leq p_i\} \\ \subseteq& \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{Z}^{d+1} : 0 \leq x_l \leq h_l(p_i) \text{ for each } 1 \leq l \leq d,\ x_i = n,\ k \leq x_{d+1} \leq p_i\}.\end{aligned}$$

Therefore,

$$|\Delta_i(p_i)| \leq \frac{p_i - k + 1}{h_i(p_i) + 1}\prod_{l=1}^{d}(h_l(p_l) + 1).$$

If $k \leq \eta < p_i$, then

$$h_i(\eta) + 1 \leq h_i(p_i - 1) + 1 = h_i(p_i) = n \leq f_i(k) \leq f_i(\eta).$$

And then

$$h_i(\eta) = h_i(\eta - 1) + 1.$$

Therefore,

$$h_i(p_i) - h_i(k) = p_i - k.$$

Such

$$|\Delta_i(p_i)| \leq \frac{h_i(p_i) - h_i(k) + 1}{h_i(p_i) + 1} \prod_{l=1}^{d} (h_l(p_i) + 1) \leq \prod_{l=1}^{d} (h_l(p_i) + 1).$$

So that for any $m \geq 0$, if $m \in \{p_i(n) : n \geq 1\}$, then

$$|\Delta_i(m)| \leq \prod_{l=1}^{d} (h_l(m) + 1). \tag{2.13}$$

Obviously, (2.13) is true for $m = 0$ since $\Delta_i(0) = \{(0, \cdots, 0)\}$. Notice that $p_i(0) = 0$ and the fact that if $m \in \mathbb{Z} \backslash \{p_i(n) : n \geq 0\}$ then $\Delta_i(m) = \emptyset$. Therefore, (2.13) are true for all $m \geq 0$. Finally, for any $m \geq 0$

$$|\partial_m| \leq \sum_{i=1}^{d+1} |\Delta_i(m)| \leq \sum_{i=1}^{d+1} \prod_{l=1}^{d} (h_l(m) + 1) \leq (d+1) \prod_{i=1}^{d} (h_i(m) + 1).$$

We have completed the proof of the lemma. □

3 Proof of Theorem 1.1

We shall use the notation of electric network. Every edge of Wedge$(f_1, \cdots, f_d)$ is assigned a unit conductance. So that, we get an electric network. For sets $A, B \subset \mathbb{V}$ with $A \cap B = \emptyset$, denote by $\mathcal{R}(A \leftrightarrow B)$ the effective resistance between A and B in the electric network. For simplicity, we label O as the origin of $\mathbb{Z}^{d+1}$ and set

$$\mathbb{V}_r = \bigcup_{n=0}^{r} \partial_r$$

for each $r \geq 1$. Then we have the following lemma.

Lemma 3.1 *For each $r \geq 1$*

$$\mathcal{R}(O \leftrightarrow \partial_r) \geq \frac{1}{2(d+1)^2} \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}.$$

Proof. Notice that $\partial_0 = \{O\}$. By Lemma 2.2, for each $n \geq 1$ the neighbor of ∂_n are ∂_{n-1} and ∂_{n+1} in Wedge$(f_1, \cdots, f_d)$. So that ∂_n is a cutset which separates O from ∂_{n+s}. The rest proof is easy and one can refer to [9]. Fix r. The effective resistance from O to ∂_r in $(\mathbb{V}, \mathbb{E})$ is equal to that in its subgraph with vertex set $\mathbb{V}_r$. We short together all the vertices in ∂_n for each $0 \leq n \leq r$. And replace the edges between ∂_n and ∂_{n+1} by a single edge of resistance $\frac{1}{b_n}$, where b_n is the number of edges connect ∂_n with ∂_{n+1}. This new network is a series network with the same effective resistance from O to ∂_r. Thus, Rayleigh's monotonicity law shows that the effective resistance from O to ∂_r in $\mathbb{V}_r$ is at least $\sum_{n=0}^{r-1} \frac{1}{b_n}$. By Lemma 2.3 and the fact that every vertex of Wedge$(f_1, \cdots, f_d)$ has at most $2(d+1)$ neighbor,

$$\mathcal{R}(O \leftrightarrow \partial_r) \geq \sum_{n=0}^{r-1} \frac{1}{b_n} \geq \frac{1}{2(d+1)} \sum_{n=0}^{r-1} \frac{1}{|\partial_n|} \geq \frac{1}{2(d+1)^2} \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{h_i(n)+1}.$$

□

On the other hand we can estimate the upper bound of $\mathcal{R}(x \leftrightarrow \partial_r)$.

Lemma 3.2 *There exists $C_d > 0$ which depends only on d such that for any $r \geq 1$ and any $x \in \mathbb{V}_{r-1}$,*

$$\mathcal{R}(x \leftrightarrow \partial_r) \leq C_d \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{h_i(n)+1}.$$

Proof. Outline of the proof. We shall construct $2d$ functions $g_{\pm i}(\cdot)$ first. These functions will help us to find a subset $\mathbb{V}_x$ which satisfies that $x \in \mathbb{V}_x \subseteq \mathbb{V}_r$. Such $\mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x)$, the resistance between x and $\Delta_{d+1}(r) \cap \mathbb{V}_x$ in the subgraph with vertex set $\mathbb{V}_x$, is greater than $\mathcal{R}(x \leftrightarrow \partial_r)$. Furthermore, we show the relation between $\mathbb{V}_x$ and Wedge$(h_1, \cdots, h_d)$. As known from Lyons[8], the related resistance in Wedge$(h_1, \cdots, h_d)$ can be gotten. So do $\mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x)$.

Fix $x = (x_1, \cdots, x_d, s) \in \mathbb{V}_{r-1}$. We shall construct $2d$ nonnegative integer valued functions on $\mathbb{Z}^+$. Fix $1 \leq i \leq d$. First set

$$g_{\pm i}(0) = x_i.$$

Suppose that the definition of $g_{\pm i}(n)$ is known, we define $g_{\pm i}(n+1)$ in three cases.
(1) If $h_i(n+1) = h_i(n)$, then we set $g_{\pm i}(n) = g_{\pm i}(n+1)$.
(2) If $h_i(n+1) = h_i(n) + 1$ and if $g_{-i}(n) = 0$, then we set $g_{-i}(n+1) = 0$ and $g_i(n+1) = g_i(n) + 1$.
(3) Otherwise, if $h_i(n+1) = h_i(n) + 1$ and if $g_{-i}(n) > 0$, then we set $g_{-i}(n+1) = g_{-i}(n) - 1$ and $g_i(n+1) = g_i(n)$.
We say that these functions $g_{\pm i}(n)$ has the properties (a),(b) and (c). Where

(a) : $g_i(n+1) - g_i(n) \in \{0, 1\}$ and $g_{-i}(n+1) - g_{-i}(n) \in \{0, -1\}$ for each $n \geq 0$;

(b) : $g_i(n) - g_{-i}(n) = h_i(n)$ for each $n \geq 0$;

(c) : $0 \leq g_{-i}(n) \leq g_i(n) \leq \min\{f_i(n+s), h_i(r)\}$ for each $0 \leq n \leq r - s$.

Obviously, (a) are true for all $n \geq 0$. Next we shall prove (b) by induction to n. It is true for $n = 0$ since $h_i(0) = 0$. Suppose (b) is true for $n = m$ and we shall check $n = m + 1$. In any case of (1),(2) and (3), there has

$$h_i(m+1) - h_i(m) = [g_i(m+1) - g_i(m)] - [g_{-i}(m+1) - g_{-i}(m)].$$

By the assumption that (b) is true for $n = m$, we can get that (b) is still true for $n = m+1$. Such (b) is true for any $n \geq 0$. Again we prove (c) by induction. Owing to $x \in \mathbb{V}_{r-1}$ and $x_{d+1} = s$,

$$0 \leq x_i \leq h_i(x_{d+1}) = h_i(s) \leq \min\{h_i(r), f_i(s)\}.$$

So (c) is true for $n = 0$. Suppose (c) is true for $n = m < r - s$ and we shall check $n = m + 1$.
If (1) is true for $n = m + 1$, then by the assumption that (c) is true for $n = m$ and the monotone property of $f_i(\cdot)$, we have (c) for $n = m + 1$.
If (2) is true for $n = m + 1$, then what we need to care is only $g_i(n+1)$. However, by the result (b) we have proved

$$g_i(n+1) = h_i(n+1) + g_{-i}(n+1) = h_i(n+1) \leq f_i(n+1) \leq f_i(s+n+1).$$

Furthermore, since $n < r - s$,

$$h_i(n+1) \leq h_i(r).$$

Therefore (c) is true for $n = m + 1$.
If (3) is true for $n = m + 1$, then what we need to care is only $g_{-i}(n + 1)$. But by the condition that $g_i(n) > 0$, we have

$$g_{-i}(n + 1) = g_{-i}(n) - 1 \geq 0.$$

Hence (c) is true, too. Therefore, in any case (c) is true for $n = m+1$ with $n < r-s$.

As a result, we can define vertex set $\mathbb{V}_x$ and edge set $\mathbb{E}_x$. Let

$$\mathbb{V}_x = \{(u_1, \cdots, u_d, n+s) \in \mathbb{Z}^{d+1} : 0 \leq n \leq r-s,\ g_{-i}(n) \leq u_i \leq g_i(n) \text{ for each } 1 \leq i \leq d\}.$$

Let

$$\mathbb{E}_x = \{[u, v] \in \mathbb{E} : u, v \in \mathbb{V}_x\}.$$

The definition does not make confusion of $\mathbb{V}_x$ and $\mathbb{V}_n$ since x is a vector. By (c),

$$x \in \mathbb{V}_x \subseteq \mathbb{V}_r.$$

Hence graph $(\mathbb{V}_x, \mathbb{E}_x)$ is a subgraph of Wedge$(f_1, \cdots, f_d)$. Notice that

$$\partial_r \cap \mathbb{V}_x \supseteq \Delta_{d+1}(r) \cap \mathbb{V}_x.$$

(Actually $\partial_r \cap \mathbb{V}_x = \Delta_{d+1}(r) \cap \mathbb{V}_x$, but we omit the proof here since it is irrelevant to our main result.) By the Rayleigh's monotonicity law, the effective resistance between x and $\Delta_{d+1}(r) \cap \mathbb{V}_x$ in the subgraph is greater than that in the old graph. That is,

$$\mathcal{R}(x \leftrightarrow \partial_r) \leq \mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x). \tag{3.1}$$

So that we need only to estimate the upper bound of $\mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x)$.

We shall show the relation between $(\mathbb{V}_x, \mathbb{E}_x)$ and Wedge$(h_1, \cdots, h_d)$. Let

$$\mathbb{H} = \{(x_1, \cdots, x_d, n) \in \mathbb{Z}^{d+1} : 0 \leq x_i \leq h_i(n) \text{ for each } 1 \leq i \leq d, 0 \leq n \leq r - s\}.$$

Obviously, $\mathbb{H}$ is a subset of vertices of Wedge$(h_1, \cdots, h_d)$. By the construction of $g_{\pm i}(\cdot)$, one can easily check that there has for each $n \geq 1$

$$\text{either} \quad g_{-i}(n) = g_{-i}(n-1) \quad \text{or} \quad g_i(n) = g_i(n-1).$$

So we can define

$$L_i(n) = \min\{g_{si}(n) : g_{si}(n) = g_{si}(n-1), s \in \{-1, 1\}\}.$$

Let $\Gamma(x) = O$. For each $(u_1, \cdots, u_d, n+s) \in \mathbb{V}_x$ with $n \geq 1$, let

$$\Gamma(u_1, \cdots, u_d, n+s) = (|u_1 - L_1(n)|, \cdots, |u_d - L_d(n)|, n).$$

By (b), Γ is a bijection function from $\mathbb{V}_x$ to $\mathbb{H}$. Obviously, $[u, v] \in \mathbb{E}_x$ if and only if $[\Gamma(u), \Gamma(v)]$ is an edge of Wedge$(h_1, \cdots, h_d)$ for each pair of u and v with $u_{d+1} = v_{d+1}$. Moreover, for any $u \in \mathbb{V}_x$ we have that $u - e_{d+1} \in \mathbb{V}_x$ if and only if $\Gamma(u) - e_{d+1} \in \mathbb{H}$.

Since $h_i(\cdot)$ increases at most one at each step, we can use the result of Lyons[8]. That is, there exists a unit flow $\mathbf{w}$ from O to $\Delta_{d+1}(r-s)$ in the subgraph of Wedge$(h_1, \cdots, h_d)$ with vertex set $\mathbb{H}$, such that for each $u \in \mathbb{H}$ with $u_{d+1} = n < r-s$,

$$\mathbf{w}(u, u + e_{d+1}) = \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}, \tag{3.2}$$

and the energy of $\mathbf{w}$ has upper bound

$$\mathcal{E}(\mathbf{w}) \leq C_d \sum_{n=0}^{r-s-1} \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}, \tag{3.3}$$

where $C_d < \infty$ and depends only on d. Let $\mathbf{w}_x$ be a function on $\mathbb{E}_x$ and satisfies that for each $[u, v] \in \mathbb{E}_x$ with $u_{d+1} = v_{d+1}$,

$$\mathbf{w}_x(u, v) = \mathbf{w}(\Gamma(u), \Gamma(v)).$$

and for each $u \in \mathbb{V}_{r-1}$ with $u_{d+1} = n$, let

$$\mathbf{w}_x(u, u + e_{d+1}) = \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}.$$

Directly calculate

$$\sum_{v:[u,v] \in \mathbb{E}_x} \mathbf{w}_x(u, v)$$

$$
\begin{aligned}
&= \mathbf{w}_x(u, u+e_{d+1}) + \mathbf{w}_x(u, u-e_{d+1}) 1_{\{u-e_{d+1} \in \mathbb{V}_x\}} + \sum_{v:[u,v]\in \mathbb{E}_x, u_{d+1}=v_{d+1}} \mathbf{w}_x(u,v) \\
&= \prod_{i=1}^{d} \frac{1}{h_i(n)+1} - \prod_{i=1}^{d} \frac{1}{h_i(n-1)+1} 1_{\{u-e_{d+1}\in \mathbb{V}_x\}} + \sum_{v:[u,v]\in \mathbb{E}_x, u_{d+1}=v_{d+1}} \mathbf{w}(\Gamma(u), \Gamma(v)) \\
&= \mathbf{w}(\Gamma(u), \Gamma(u)+e_{d+1}) + \mathbf{w}(\Gamma(u), \Gamma(u)-e_{d+1}) 1_{\{\Gamma(u)-e_{d+1}\in \mathbb{H}\}} + \sum_{z\in \mathbb{H}:\|u-z\|_1=1, u_{d+1}=z_{d+1}} \mathbf{w}(\Gamma(u), z) \\
&= \sum_{z\in\mathbb{H}:\|u-z\|_1=1} \mathbf{w}(\Gamma(u), z).
\end{aligned}
$$

Together with the fact that $\mathbf{w}$ is a unit flow, we get that $\mathbf{w}_x$ is a unit flow from x to $\Delta_{d+1}(r) \cap \mathbb{V}_x$ in graph $(\mathbb{V}_x, \mathbb{E}_x)$. Obviously

$$
\mathcal{E}(\mathbf{w}_x) = \mathcal{E}(\mathbf{w}). \tag{3.4}
$$

Together (3.1), (3.3) and (3.4), we have

$$
\mathcal{R}(x \leftrightarrow \partial_r) \le \mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r)\cap \mathbb{V}_x) \le \mathcal{E}(\mathbf{w}_x) = \mathcal{E}(\mathbf{w}) \le C_d \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{h_i(n)+1}.
$$

□

Proof of Theorem 1.1. As it is well known, a connect graph with local finite degree is recurrent if and only if the resistance from any one vertex to the infinity in the graph is infinite (Refer to [9], Proposition 9.1). Together with Lemmas 3.1 and 3.2, we have the desired result. □

4 Proof of Theorem 1.2

Lemma 4.1 *Let G be a graph of bounded degrees with a distinguished vertex o and suppose that there exists a sequence of sets $(B_r)_r$ growing with r and satisfying*

$$
g_{B_r}(o,o) \to \infty \text{ as } r \to \infty \text{ and } g_{B_r}(x,x) \le C g_{B_r}(o,o), \quad \forall x \in G,
$$

for a uniform constant $C > 0$. Here, $g_B(\cdot,\cdot)$ is the green function of the simple random walk on G killed when it exits B. Then the graph G has the infinite collision property.

Proof. Refer to [2]. $\square$

Proof of Theorem 1.2. First suppose Wedge$(f_1, \cdots, f_d)$ is not a recurrent graph. Then Wedge$(f_1, \cdots, f_d)$ is a transient graph. It implies that $g_{\mathbb{V}}(O, O)$, the expected number of returning to O, is finite. One can easily get that the expected number of collisions between two independent simple random walks starting from O is less than $2(d+1)g_{\mathbb{V}}(O, O)$. So that, almost surely the number of collisions is finite. Hence, Wedge$(f_1, \cdots, f_d)$ has the finite collision property.

On the other hand, suppose Wedge$(f_1, \cdots, f_d)$ is recurrent. By Theorem 1.1 we have (1.2). Furthermore, by Lemma 3.1

$$\lim_{r\to\infty} \mathcal{R}(O \leftrightarrow \partial_r) \geq \lim_{r\to\infty} \frac{1}{2(d+1)^2} \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{f_i(n)+1} = \infty.$$

As it is known to all (refer to [2]) that for each $r \geq 1$

$$\mathcal{R}(O \leftrightarrow \partial_{r+1}) = g_{\mathbb{V}_r}(O, O).$$

So $\lim_{r\to\infty} g_{\mathbb{V}_r}(O, O) = \infty$. By Lemmas 3.1 and 3.2, for all $r \geq 1$ and $x \in$ Wedge$(f_1, \cdots, f_d)$

$$g_{\mathbb{V}_r}(x, x) \leq 2(d+1)^2 C_d \; g_{\mathbb{V}_r}(O, O).$$

By Lemma 4.1, Wedge$(f_1, \cdots, f_d)$ has the infinite collision property. $\square$

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